

# Time Reversal Problem of Heat Conduction for Elliptic Cylindrical Shell with Internal Heat Source

S. D. Bagde and N. W. Khobragade

Abstract- This paper is concerned with inverse thermoelastic problem of an elliptical plate to determine the temperature distribution and unknown temperature gradient at point  $\xi = b$  for all time ( $t > 0$ ) with the help of Mathieu transform and integral transform techniques.

Key Words- Inverse thermoelastic problem, Mathieu transform, Marchi-Fasulo transform.

AMS SUBJECT CLASSIFICATION NO. 35-XX, 44-XX, 80-XX

## 1. INTRODUCTION

The determination of initial temperature distribution from a known physically realizable temperature distribution at any time and position is known as **time reversal problem**. This type of problem has advantage of finding the temperature distribution at prior state when the temperature distribution at any position is known at any instant. In recent years time reversal problem for the circular boundary have been worked out by many authors.

**Masket** [3] has applied Green functions. **Sabherwal** [4] has used Hankel transform. **Mehta** [2] has applied Marchi and Zgrablich transform [5].

Time reversal problem of heat conduction for an elliptical cylindrical shell with heat generation and radiation condition does not seem to have been worked out so far. In the present paper we have applied finite Mathieu transform [1] and finite Marchi-Fasulo transform techniques to solve the problem with the given boundary conditions.

## 2. Required Results

### Finite Mathieu Transforms

**Gupta** [7] and **Choubey** [1] have defined the finite Mathieu transform and its properties. If the function  $T(\xi, \eta)$  satisfies Dirichlet's conditions in the region  $a \leq \xi \leq b$ ,  $0 \leq \eta \leq 2\pi$  and satisfies the radiation type boundary conditions at the boundaries  $a$  and  $b$ , then its finite Mathieu transform is defined as

$$\bar{T}(q_{n,m}) = \int_a^b \int_0^{2\pi} (\cosh 2\xi - \cos 2\eta) S_{n,m}(k_1, k_2, \xi, \eta, q_{n,m}) d\xi d\eta \quad (1)$$

Where the kernel  $S_{n,m}(k_1, k_2, \xi, \eta, q_{n,m})$  is given by

$$S_{n,m}(k_1, k_2, \xi, \eta, q_{n,m}) = C e_n(\xi, q_{n,m}) c e_n(\eta, q_{n,m}) [Fey_n(k_1, a, \eta, q_{n,m}) + Fey_n(k_2, b, \eta, q_{n,m}) - Fey_n(\xi, q_{n,m}) c e_n(\eta, q_{n,m}) C e_n(k_1, a, \eta, q_{n,m}) + c e_n(k_2, b, \eta, q_{n,m})] \quad (2)$$

And  $q_{n,m}$  are roots of

$$c e_n(k_1, a, \eta, q_{n,m}) Fey_n(k_2, b, \eta, q) - c e_n(k_2, b, \eta, q) Fey_n(k_1, a, \eta, q) = 0 \quad (3)$$

Where  $c e_n(\eta, q)$  and  $C e_n(\xi, q)$  are ordinary Mathieu function and modified Mathieu functions of first kind.  $Fey_n(\xi, q)$  is modified Mathieu function of second kind.

### Inversion of Finite Mathieu Transform

If  $T(\xi, \eta)$  satisfies Dirichlet conditions in  $a \leq \xi \leq b$ ,  $0 \leq \eta \leq 2\pi$  and  $\bar{T}(q_{n,m})$  exist then

$$T(\xi, \eta) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{\bar{T}(q_{n,m}) S_{n,m}(k_1, k_2, \xi, \eta, q_{n,m})}{C_{n,m}} \quad (4)$$

Where

$$C_{n,m} = \int_a^b \int_0^{2\pi} S_{n,m}^2(k_1, k_2, \xi, \eta, q_{n,m}) (\cosh 2\xi - \cos 2\eta) d\xi d\eta \quad (5)$$

Summation being taken over the positive roots of (3).

## 3. STATEMENT OF PROBLEM

### Heat Flow In An Elliptic Cylindrical Shell Of Finite Height With Internal Heat Source

Consider the heat conduction equation in a elliptic cylindrical shell ( $a \leq \xi \leq b$ ,  $0 \leq \eta \leq 2\pi$ ),  $-h \leq z \leq h$  when there are sources of heat within it which lead to temperature distribution. If we assume that the rate of generation of heat is independent of the temperature and the length of shell is finite, then the fundamental differential equation in elliptical co-ordinates is given by [6]

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$$\frac{1}{\alpha} \frac{\partial T}{\partial t} = \frac{2d^{-2}}{(\cosh 2\xi - \cos 2\eta)} \left( \frac{\partial^2 T}{d\xi^2} + \frac{\partial^2 T}{d\eta^2} \right) + \frac{\partial^2 T}{dz^2} + \Phi(\xi, \eta, z, t) \quad (6)$$

Where  $\alpha$  is constant known as diffusivity,  $2d$  = interfocal length of ellipse and temperature distribution,  $T$  is function of  $\xi, \eta$  and  $t$  the time function.  $\Phi(\xi, \eta, z, t)$  is taken for heat generation.

The boundary conditions are:

$$T(a, \eta, z, t) + k_1 \frac{\sqrt{2}}{\eta} (\cosh 2\xi - \cos 2\eta) \frac{\partial T(a, \eta, z, t)}{\partial \xi} = F_a(\eta, z, t) \quad (7)$$

$$t > 0, \quad 0 \leq \eta \leq 2\pi, \quad -h \leq z \leq h$$

$$T(b, \eta, z, t) + k_2 \frac{\sqrt{2}}{\eta} (\cosh 2\xi - \cos 2\eta) \frac{\partial T(b, \eta, z, t)}{\partial \xi} = F_b(\eta, z, t) \quad (8)$$

$$t > 0, \quad 0 \leq \eta \leq 2\pi, \quad -h \leq z \leq h$$

Where  $k_1, k_2$  are radiation constants on the two surfaces.

$$T(\xi, \eta, z, t) = f(\xi, \eta, z); \quad (\text{known}) \quad \text{for all } a \leq \xi \leq b$$

$$0 \leq \eta \leq 2\pi \quad (9)$$

$$T(\xi, \eta, z, 0) = g(\xi, \eta, z); \quad (\text{unknown}) \quad \text{for all } a \leq \xi \leq b$$

$$0 \leq \eta \leq 2\pi \quad (10)$$

The equations (1) –(10) constitute the mathematical formulation of the problem under consideration.

#### 4. SOLUTION OF THE PROBLEM

By applying finite Mathieu transform defined in (1) and using the conditions (7), (8) and the property of Mathieu transform given by Choubey [1], one obtains

$$\frac{dT}{dt} - k\lambda_{2n,m}^{-2} = \psi(q_{n,m}t) \quad (11)$$

Where

$$\psi(q_{n,m}t) = \bar{\Phi}(q_{n,m}t) - \int_0^{2\pi} \left[ \frac{\alpha}{k_2} S_{n,m}(k_1, k_2, q_{n,m}b) F_b(\eta, t) \frac{h}{\sqrt{2}} (\cosh 2b - \cos 2\eta)^{1/2} - \frac{\alpha}{k_1} S_{n,m}(k_1, k_2, q_{n,m}a) F_a(\eta, t) \frac{h}{\sqrt{2}} (\cosh 2a - \cos 2\eta)^{1/2} \right] d\eta \quad (12)$$

And  $\bar{\Phi}(q_{n,m}t), \bar{T}(q_{n,m}t)$  are finite Mathieu Transform of  $\phi(\xi, \eta, t)$  and  $T(\xi, \eta, t)$  defined as follows:

$$\bar{\Phi}(q_{n,m}, z, t) = \int_0^{2\pi} \int_a^b \Phi(\xi, \eta, z, t) S_{n,m}(k_1, k_2, q_{n,m}, \xi, \eta) (\cosh 2\xi - \cos 2\eta) d\xi d\eta$$

$$\bar{T}(q_{n,m}, z, t) = \int_0^{2\pi} \int_a^b T(\xi, \eta, z, t) S_{n,m}(k_1, k_2, q_{n,m}, \xi, \eta) (\cosh 2\xi - \cos 2\eta) d\xi d\eta \quad (13)$$

The solution of equation (11) is

$$\bar{T}(q_{n,m}, z, t) = e^{-\alpha\lambda_{n,m}^2 t} \int_0^1 \psi(q_{n,m}t^1) e^{\alpha\lambda_{n,m}^2 t^1} dt^1 + Ae^{-\alpha\lambda_{n,m}^2 t} \quad (14)$$

Where  $A$  is an arbitrary constant. Using the condition (5) and (6)

in equation (14), we obtain

$$\bar{g}(q_{n,m}) = \bar{f}(q_{n,m}) e^{-\alpha\lambda_{n,m}^2 t} - \int_0^1 \psi(q_{n,m}t^1) e^{-\alpha\lambda_{n,m}^2 t^1} dt^1$$

Applying the inversion finite Mathieu transform we obtain

$$g(\xi, \eta, z) = \sum_{l=0}^{\infty} \frac{P_l(z)}{\lambda_l} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{1}{C_{n,m}} \left[ \bar{f}(q_{n,m}) e^{\alpha\lambda_{n,m}^2 t} - \int_0^1 \psi(q_{n,m}t^1) e^{\alpha\lambda_{n,m}^2 t^1} dt^1 \right] \times S_{n,m}(k_1, k_2, \xi, \eta) \quad (15)$$

#### 5. CONCLUSION

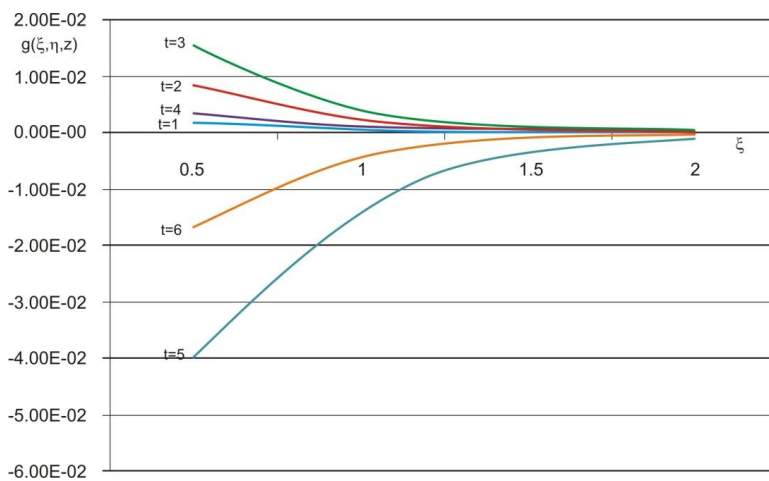
In this chapter, an elastic vibration of elliptic plate has been determined with the help of finite Mathieu transform and finite Marchi-Fasulo transform and Laplace transform techniques. The expression is represented graphically. The results that are obtained can be useful to the design of structures or machines in engineering applications.

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Graph  $g(\xi, \eta, z)$  versus  $\xi$  for different value of  $t$ .